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A SUFFICIENT CONDITION FOR THE INSTABILITY OF THE ϵ-ALGORITHM*

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In a previous treatment [1] of the q–d algorithm a sufficient condition for the instability in the construction of certain auxiliary quantities by use of this algorithm was given. The criterion was established by contrasting a special distribution of error in the initial conditions with that in the derived quantities. The special error distribution arose only as a possibility (but this in itself of course shows that a certain degree of instability can occur) and was chosen to allow certain differential forms of the q–d algorithm to be used in the analysis. At the time at which the note referred to was written, corresponding differential forms of the ϵ-algorithm had not been discovered, but this has latterly been remedied and accordingly a similar treatment of the ϵ-algorithm is possible.

If quantities $\epsilon_s^{(m)}$ $m, s = 0, 1, \ldots$ are produced by application of the ϵ-algorithm relationships [2]

$$\epsilon_{s+1}^{(m)} = \epsilon_{s-1}^{(m+1)} + \frac{1}{\epsilon_s^{(m+1)} - \epsilon_s^{(m)}} \tag{1}$$

to the initial conditions

$$\epsilon_{-1}^{(m)} = 0,\ \epsilon_0^{(m)} = \sum_{s=0}^{m-1} \phi^{(s)}(a) z^{-s-1},\ m = 1, 2, \ldots;\ \epsilon_0^{(0)} = 0 \tag{2}$$

then it may be shown, by an extension of methods similar to those employed in [3], that

$$\begin{gathered}\{\epsilon_{2s+1}^{(m)} - \epsilon_{2s-1}^{(m)}\}\{\phi(a) - z\epsilon_{2s}^{(m)} + \frac{\partial}{\partial a}\epsilon_{2s}^{(m)}\} = z \\ \{\epsilon_{2s+2}^{(m)} - \epsilon_{2s}^{(m)}\}\{z\epsilon_{2s+1}^{(m)} + \frac{\partial}{\partial a}\epsilon_{2s+1}^{(m)}\} = z\end{gathered} \tag{3}$$

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Now, in any application of the ϵ-algorithm to the initial condition

$$\epsilon_{-1}^{(m)} = 0, \epsilon_0^{(m)} = S_m \qquad m = 1, 2, \ldots, \epsilon_0^{(0)} = 0 \tag{4}$$

the quantities S_m $m = 0, 1, \ldots$ may be construed as the partia sums of the series

$$S_m = \sum_{s=0}^{m-1} \phi^{(s)}(a) \qquad m = 1, 2, \ldots \tag{5}$$

corresponding to $z = 1$ in (2).

A certain error distribution may be produced in the initial con ditions (4) by letting an increase to $a + \Delta a$; in this case the erro in $\epsilon_0^{(m')}$ is, from (5),

$$\{\epsilon_0^{(m'+1)} - \epsilon_0^{(1)}\}\Delta a \tag{6}$$

The corresponding errors in the quantities $\epsilon_{2s}^{(m)}$ are, from (3) given by

$$\left\{\epsilon_{2s}^{(m)} - \epsilon_0^{(1)} + \frac{1}{\epsilon_{2s+1}^{(m)} - \epsilon_{2s-1}^{(m)}}\right\}\Delta a \tag{7}$$

and in $\epsilon_{2s+1}^{(m)}$, by

$$\left\{\frac{1}{\epsilon_{2s+2}^{(m)} - \epsilon_{2s}^{(m)}} - \epsilon_{2s+1}^{(m)}\right\}\Delta a. \tag{8}$$

The ratios of the relative errors in $\epsilon_{2s}^{(m)}$ and $\epsilon_{2s+1}^{(m)}$ to those in $\epsilon_0^{(m')}$ which indicate the condition of the ϵ-process, are given by

$$\left\{\frac{(\epsilon_{2s}^{(m)} - \epsilon_0^{(1)})(\epsilon_{2s+1}^{(m)} - \epsilon_{2s-1}^{(m)}) + 1}{(\epsilon_{2s+1}^{(m)} - \epsilon_{2s-1}^{(m)})\epsilon_{2s}^{(m)}}\right\} \frac{\epsilon_0^{(m')}}{(\epsilon_0^{(m'+1)} - \epsilon_0^{(1)})} \tag{9}$$

$$m - 2s \leq m' \leq m; s = 0, 1, \ldots; m' = 1, 2, \ldots$$

and

$$\left\{\frac{1 - \epsilon_{2s+1}^{(m)}(\epsilon_{2s+2}^{(m)} - \epsilon_{2s}^{(m)})}{\epsilon_{2s+1}^{(m)}(\epsilon_{2s+2}^{(m)} - \epsilon_{2s}^{(m)})}\right\} \frac{\epsilon_0^{(m')}}{\epsilon_0^{(m'+1)} - \epsilon_0^{(1)}} \tag{10}$$

$$m - 2s - 1 \leq m' \leq m; s = 0, 1, \ldots; m' = 1, 2, \ldots$$

These results may be extended by the following considerations If quantities $\epsilon_s^{(m)}$ are produced by applying relationships (1) to the initial conditions (4), and quantities $\tilde{\epsilon}_s^{(m)}$ are produced by applying relationships (1) to the initial conditions

$$\tilde{\epsilon}_{-1}^{(m)} = 0, \quad \tilde{\epsilon}_0^{(m)} = b + S_m \qquad m = 1, 2, \ldots; \tilde{\epsilon}_0^{(0)} = b \tag{11}$$

then

$$\tilde{\epsilon}_{2s}^{(m)} = b + \epsilon_{2s}^{(m)}, \tilde{\epsilon}_{2s+1}^{(m)} = \epsilon_{2s+1}^{(m)} \qquad m, s = 0, 1, \ldots \tag{12}$$

That is to say that any one of the initial values $S_{m''}$ $m'' = 1, 2, \ldots$ in (4) can be made to play the rôle of $\epsilon_0^{(1)}$ in (9) and (10), merely by subtracting $\epsilon_0^{(m''-1)}$ from all the initial values. The ratios (9) and (10) then become

$$\frac{\{(\epsilon_{2s}^{(m)} - \epsilon_0^{(m'')})(\epsilon_{2s+1}^{(m)} - \epsilon_{2s-1}^{(m)}) + 1\}(\epsilon_0^{(m')} - \epsilon_0^{(m''-1)})}{(\epsilon_{2s+1}^{(m)} - \epsilon_{2s-1}^{(m)})(\epsilon_{2s}^{(m)} - \epsilon_0^{(m'-1)})(\epsilon_0^{(m'+1)} - \epsilon_0^{(m'')})} \tag{13}$$

$m - 2s \le m' \le m; s = 0, 1, \ldots; m'' = 1, 2, \ldots; m' = m'', m'' + 1, \ldots$

and

$$\frac{\{1 - \epsilon_{2s+1}^{(m)}(\epsilon_{2s+2}^{(m)} - \epsilon_{2s}^{(m)})\}(\epsilon_0^{(m')} - \epsilon_0^{(m''-1)})}{\epsilon_{2s+1}^{(m)}(\epsilon_{2s+2}^{(m)} - \epsilon_{2s}^{(m)})(\epsilon_0^{(m'+1)} - \epsilon_0^{(m'')})} \tag{14}$$

$m - 2s - 1 \le m' \le m; s = 0, 1, \ldots; m'' = 1, 2, \ldots;$
$m' = m'', m'' + 1, \ldots$

The main result of this note may thus be stated as:

If either of the expressions (13) and (14) has a modulus greater than unity the ϵ-process is unstable.

It would appear from expressions (13) and (14) that if the differences $\epsilon_0^{(m'+1)} - \epsilon_0^{(m'')}$ $m'' = 1, 2, \ldots; m' = m'', m'' + 1, \ldots$ are consistently small then the ϵ-process is likely to be unstable, but this corresponds to the case in which the ϵ-algorithm is applied to the transformation of a slowly convergent series of terms which have approximately constant modulus and argument, and in this case it is a matter of numerical experience that the ϵ-process is unstable.

REFERENCES

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[2] P. Wynn, On a Device for Computing the $e_m(S_n)$ Transformation, MTAC, X, (1956), 91–96.

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